7.4 Proving Trigonometric Identities

A Equations versus Identities
An equation is a mathematical statement which is true if the variable has only specific values. To solve an equation means to find the set of values of the variable that satisfy the equation.

An identity is a mathematical statement which is true for any values of the variable. To prove an identity means to start with a side and apply algebraic rules until the other side is reached.

Ex 1. Classify the following expressions as identity or equation:

a) $x^2 - 4 = 0$

b) $(x+2)(x-2) = x^2 - 4$

B Pythagorean Identities

The following identities are called Pythagorean Identities:

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

C Identities based on Definitions

The following definitions might be used when proving trigonometric identities:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$
$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

Ex 2. Prove the following identity

$1 - \cos^2 x = \cos^2 x \tan^2 x$

Ex 3. Prove the following identity

$\frac{\sin x}{1 - \cos x} = \frac{1 + \cos x}{\sin x}$

Ex 4. Prove the following identity

$\frac{1}{1 + \sin x} + \frac{1}{1 - \sin x} = 2\sec^2 x$

Ex 5. Prove the following identity

$\frac{1 + \cos 2x}{\sin 2x} = \cot x$

Reading: Nelson Textbook, Pages 412-416

Homework: Nelson Textbook, Page 417: #8, 9, 10, 11, 16

More Trigonometric Identities

Part A

1. $\tan x \cos x = \sin x$
2. $\cot x \sec x = \csc x$
3. $\sin x \cot x = \cos x$
4. $\tan x \csc x = \sec x$
5. $\sin x = \dfrac{\tan x}{\sec x}$
6. $\dfrac{\cot x}{\csc x} = \cos x$
7. $(1+\sin x)\csc x = 1+\csc x$
8. $(1+\csc x)\sin x = 1+\sin x$
9. $(\sec x - 1)\cos x = 1-\cos x$
10. $\sin x \sec x \cot x = 1$
11. $\dfrac{1-\tan x}{1-\cot x} = -\tan x$
12. $\cot x = \dfrac{1+\cot x}{1+\tan x}$
13. $\sin x \tan x + \sec x = \dfrac{\sin^2 x + 1}{\cos x}$
14. $\dfrac{1+\cos x}{1-\cos x} = \dfrac{1+\sec x}{\sec x - 1}$
15. $\dfrac{1+\sin x}{1-\sin x} = \dfrac{\csc x + 1}{\csc x - 1}$
16. $\dfrac{1+\tan x}{1+\cot x} = \dfrac{1-\tan x}{\cot x - 1}$
17. $\dfrac{1+\sin x}{1+\csc x} = \sin x$
18. $\dfrac{\sin x + \tan x}{\cos x + 1} = \tan x$
19. $\sin^2 x \cot^2 x = 1-\sin^2 x$
20. $\csc^2 x - 1 = \csc^2 x \cos^2 x$
21. $\sin^2 x = \dfrac{\tan^2 x}{1+\tan^2 x}$
22. $\dfrac{\sin x + \cos x \cot x}{\cot x} = \sec x$
23. $\dfrac{\cos x \tan x}{\csc x} = 1-\cos^2 x$
24. $\dfrac{\cos x}{1+\sin x} + \dfrac{\cos x}{1-\sin x} = 2\sec x$
25. $\dfrac{\sin x + \cos x}{\csc x + \sec x} = \sin x \cos x$
26. $\dfrac{\tan x}{\sec x + 1} = \dfrac{\sec x - 1}{\tan x}$
27. $\dfrac{1}{1+\sin x} + \dfrac{1}{1-\sin x} = 2\sec^2 x$
28. $(1+\cot^2 x)\tan^2 x = \sec^2 x$
29. $(1-\cos^2 x)(1+\tan^2 x) = \tan^2 x$
30. $\tan x + \cot x = \sec x \csc x$
31. $\sec^2 x + \csc^2 x = \sec^2 x \csc^2 x$
32. $\sec^2 x + \csc^2 x = (\tan x + \cot x)^2$
33. $\sin^2 x = \cos x(\sec x - \cos x)$
34. $\tan x + \tan^3 x = \dfrac{\sec^2 x}{\cot x}$
35. $\dfrac{1+\csc x}{\cot x} - \sec x = \tan x$
36. $\dfrac{(1-\cos^2 x)(\sec^2 x - 1)}{\cos^2 x} = \tan^4 x$
37. $(1+\sin x)\sec x = (1+\csc x)\tan x$
38. $\csc x - \dfrac{\cot x}{\sec x} = \sin x$
39. $\cos^2 x = (\csc x - \sin x)\sin x$
40. $\sec^2 x - 1 = (\sin x \sec x)^2$
41. $\cos^2 x = \dfrac{\cot^2 x}{1+\cot^2 x}$
42. $\dfrac{\sin x}{1-\cos x} + \dfrac{\sin x}{1+\cos x} = 2\csc x$
43. $\sin x \cot^2 x + \cos x \tan^2 x = \dfrac{\sin^3 x + \cos^3 x}{\sin x \cos x}$
44. $\dfrac{\tan^2 x + 1}{\cot^2 x + 1} = \dfrac{1-\cos^2 x}{\cos^2 x}$
45. $2\cos^2 x - 1 = \cos^4 x - \dfrac{1}{\csc^4 x}$
46. $\sin^4 x - \cos^4 x = 2\sin^2 x - 1$
47. $\dfrac{\csc x}{\sec^2 x} = \csc x - \sin x$
48. $\dfrac{\sin x + \tan x}{1+\cos x} = \tan x$
49. $\dfrac{\cos x}{1-\sin x} + \dfrac{\cos x}{1+\sin x} = 2\sec x$
50. $\dfrac{1+\sin x + \cos x}{1-\sin x + \cos x} = \dfrac{1+\sin x}{\cos x}$
51. $\dfrac{1-\sin x}{1+\sin x} = (\tan x - \sec x)^2$

Part B

1. $1+\sin 2x=(\sin x+\cos x)^2$

2. $\sin 2x=2\cot x\sin^2 x$

3. $\cos 2x=\dfrac{1-\tan^2 x}{1+\tan^2 x}$

4. $\sec^2 x=\dfrac{2}{1+\cos 2x}$

5. $\dfrac{1-\cos 2x}{2}=\sin^2 x$

6. $\dfrac{\sin^2 x+\cos^2 x}{\sin^2 x-\cos^2 x}=-\sec 2x$

7. $\dfrac{(\sin x+\cos x)^2}{\sin 2x}=1+\csc 2x$

8. $\dfrac{\sin 2x}{1-\cos 2x}=\dfrac{1}{\tan x}$

9. $\dfrac{\cos 2x}{\sin 2x+1}=\dfrac{1-\tan x}{1+\tan x}$

10. $\cot\dfrac{\alpha}{2}=\dfrac{1+\cos\alpha}{\sin\alpha}$

11. $\tan\dfrac{\alpha+\beta}{2}=\dfrac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta}$

12. $\cos(a+b)\cos(a-b)=\cos^2 b-\sin^2 a$

13. $\sin^2\alpha\cos^2\alpha=\dfrac{1-\cos 4\alpha}{8}$

14. $\sin x\sin y=\dfrac{\cos(x-y)-\cos(x+y)}{2}$

15. $\cos x\cos y=\dfrac{\cos(x-y)+\cos(x+y)}{2}$

16. $\sin x\cos y=\dfrac{\sin(x-y)+\sin(x+y)}{2}$

17. $\sin x+\sin y=2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$

18. $\cos x+\cos y=2\cos\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$

19. $\cos x-\cos y=-2\sin\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$

20. $\sin x-\sin y=2\cos\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$

21. $\cos u+\sin v=\left(\cos\dfrac{u-v}{2}-\sin\dfrac{u-v}{2}\right)\left(\cos\dfrac{u+v}{2}+\sin\dfrac{u+v}{2}\right)$

22. $\cos u-\sin v=\left(\cos\dfrac{u+v}{2}-\sin\dfrac{u+v}{2}\right)\left(\cos\dfrac{u-v}{2}+\sin\dfrac{u-v}{2}\right)$

23. $\cos(a-b)\cos(t+u)-\cos(a+b)\cos(t-u)=\sin(u+a)\sin(b-t)-\sin(u-a)\sin(b+t)$

Part C

1. If $x+y+z=\pi$ prove that $\sin 2x+\sin 2y+\sin 2z=4\sin x\sin y\sin z$.

2. If $x+y+z=\pi$ prove that $\tan x+\tan y+\tan z=\tan x\tan y\tan z$.

3. If $x+y+z=\pi/2$ prove that $\cot x+\cot y+\cot z=\cot x\cot y\cot z$.

Part D

1. $\sin\alpha+\sin(\alpha+\varepsilon)+\sin(\alpha+2\varepsilon)+\ldots+\sin(\alpha+n\varepsilon)=\dfrac{\sin\dfrac{(n+1)\varepsilon}{2}\sin\left(\alpha+\dfrac{n\varepsilon}{2}\right)}{\sin\dfrac{\varepsilon}{2}}$

2. $\cos\alpha+\cos(\alpha+\varepsilon)+\cos(\alpha+2\varepsilon)+\ldots+\cos(\alpha+n\varepsilon)=\dfrac{\sin\dfrac{(n+1)\varepsilon}{2}\cos\left(\alpha+\dfrac{n\varepsilon}{2}\right)}{\sin\dfrac{\varepsilon}{2}}$

Hint: Multiply both sides by $\sin\dfrac{\varepsilon}{2}$ and use the identities **14-20** from **Part B**.

